from Hmk $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ partial fraction

$$= \int_{-\infty}^{\infty} \left(\frac{c_1}{x+i} + \frac{c_2}{x-i} \right) dx$$

$$= \lim_{R\to\infty} \int_{-R}^{R} \left(\frac{c_1}{x+i} + \frac{c_2}{x-i} \right) dx$$

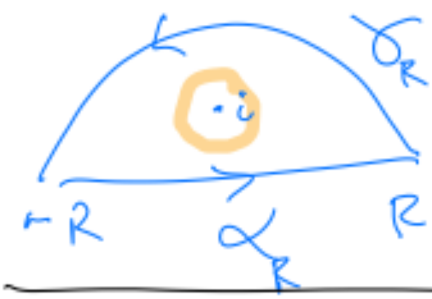
Also

$$\lim_{R\to\infty} \left| \int_\gamma \frac{1}{z^2+1}\,dz \right| = 0 \quad \text{(show).}$$

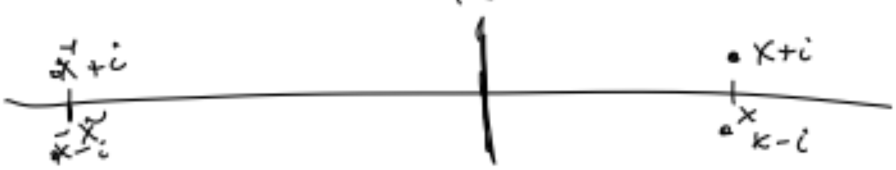
$\Rightarrow$ Answer is

$$\lim_{R\to\infty} \int_{\alpha_R \cup \gamma_R} \left(\frac{c_1}{z+i} + \frac{c_2}{z-i} \right) dz$$

holomorphic ~ def. thm $\to$ fundamental integral.



Alternately $\displaystyle\int_{-R}^{R} \frac{c_1}{x+i}\,dx + \int_{-R}^{R} \frac{c_2}{x-i}\,dx$



so principal branch works.

Example Compute $I = \int_0^{\pi} \frac{\cos(2\theta)}{1+a^2-2a\cos\theta}\,d\theta$, where $0<a<1$

since $\cos(\theta)$ & $\cos(2\theta)$ are even fcns,

$$I = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos(2\theta)}{1+a^2-2a\cos\theta}\,d\theta$$

I want to integrate over S^1 = unit circle.

standard trick: On the unit circle

$$z = e^{i\theta},\quad \frac{1}{z} = \bar{z} = e^{-i\theta},\quad z^2 = e^{2i\theta},\quad \frac{1}{z^2} = e^{-2i\theta}$$

$$\Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right),\quad \cos(2\theta) = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$$

$$dz = ie^{i\theta}\,d\theta$$

$$-ie^{-i\theta}\,dz = d\theta$$

$$\Rightarrow \frac{-i\,dz}{z} = d\theta$$

$$\left[= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta\right) = \frac{1}{2}(2\cos\theta) = \cos\theta \right]$$

$$\left[\text{also } \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)\right].$$

$$\Rightarrow I = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos(2\theta)}{1+a^2-2a\cos(\theta)}\,d\theta$$

$$= \frac{1}{2}\int_{S^1} \frac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)(-i)}{1+a^2-2a\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)}\frac{dz}{z}$$

$$= \frac{1}{4}\int_{S^1} \frac{(-i)\left(z^2+\frac{1}{z^2}\right)dz}{z(1+a^2)-a(z^2+1)}$$

we just care about singularities

(When is the denom zero

$$z(1+a^2) = a(z^2+1)$$

$$\frac{(1+a^2)}{a} = \left(\frac{z^2+1}{z}\right)$$

$$\left(\frac{1}{a}+a\right) = z + \frac{1}{z} \quad ?$$

$$az^2 - (1+a^2)z + a = 0$$

Roots $z = \frac{(1+a^2) \pm \sqrt{(1+a^2)^2 - 4a^2}}{2a}$

$1+2a^2+a^4-4a^2$

$1-2a^2+a^4$

$=(1-a^2)^2$

$$z = \frac{1+a^2 \pm (1-a^2)}{2a}$$

$$z = \left(\frac{2}{2a} = \frac{1}{a}\right) \text{ or } \left(\frac{2a^2}{2a} = a\right)$$

outside unit circle

inside unit circle

$\frac{1+z^4}{z^2}$

$$I = \frac{1}{4}\int_{S^1} \frac{(+i)\left(z^2+\frac{1}{z^2}\right)}{+a\left(z-\frac{1}{a}\right)(z-a)}\,dz$$

$$= \frac{i}{4a}\int_{S^1} \overbrace{\frac{1+z^4}{z^2\left(z-\frac{1}{a}\right)(z-a)}}^{g(z)}\,dz$$

$$= \frac{i}{4a}2\pi i\left(\mathrm{Res}(g,a) + \mathrm{Res}(g,0)\right)$$

$$g(z) = \frac{1+z^4}{z^2(z-\frac{1}{a})(z-a)} = \frac{1}{z-a} \cdot \underbrace{\overbrace{\frac{1+z^4}{z^2(z-\frac{1}{a})}}^{h(z)}}_{\text{holom at } a}$$

$(c_0 + c_1(z-a) + \dots)$

$c_0 = h(a)$

$$\boxed{\operatorname{Res}(g,a) = \frac{1+a^4}{a^2(a-\frac{1}{a})}}.$$

$$g(z) = \frac{1}{z^2}\underbrace{\left(\frac{1+z^4}{z^2-(a+\frac{1}{a})z+1}\right)}$$

$$B(z) = \underset{\substack{\shortparallel \\ 1}}{\overset{B(0)}{a_0}} + \overset{B'(0)}{a_1}(z) + \overset{\frac{B''(0)}{2!}}{a_2} z^2$$

$$B(z) = \frac{1+z^4}{z^2-(a+\frac{1}{a})z+1}$$

$$B'(z) = \frac{(z^2-(a+\frac{1}{a})z+1)(4z^3) - (1+z^4)(2z-(a+\frac{1}{a}))}{\text{bottom}^2}$$

$$B'(0) = \frac{0 + 1\cdot(a+\frac{1}{a})}{1} = \left(a+\frac{1}{a}\right).$$

$$\therefore \operatorname{Res}(g,0) = a+\frac{1}{a}.$$

$$\therefore I = \frac{i}{4a} 2\pi i\left(\operatorname{Res}(g,a) + \operatorname{Res}(g,0)\right)$$

$$\Rightarrow I = \frac{-\pi}{2a}\left(\frac{1+a^4}{a^2(a-\frac{1}{a})} + a + \frac{1}{a}\right)$$

$$= \frac{-\pi}{2a}\left(\frac{1+a^4}{a(a^2-1)} + a + \frac{1}{a}\right)$$

$$= -\frac{\pi}{2a}\left(\frac{\cancel{1}+a^4}{a(a^2-1)} + \overset{a^4-\cancel{a^2}}{a^2(a^2-1)} + \cancel{a^2}-\cancel{1} \right)$$

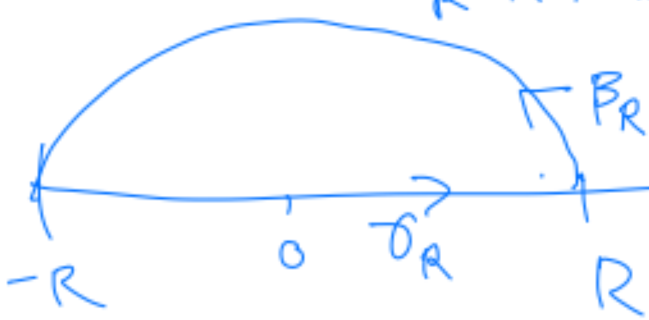
$$= \frac{-\pi}{\cancel{2a}} \frac{\cancel{2}a^{\cancel{4}\,2}}{\cancel{a}(a^2-1)} = \boxed{\frac{\pi a^2}{1-a^2}}$$

since $a<1$, this is positive.

<u>Example</u> Compute $I = \int_0^\infty \frac{\cos(x)}{x^4+5x^2+1}\,dx$.

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(x)}{x^4+5x^2+1}\,dx \qquad \text{(fcn is even)}$$

$$= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{\cos(x)}{x^4+5x^2+1}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{\gamma_R} \frac{\cos(z)}{z^4+5z^2+1}\,dz$$



$$= \mathrm{Re}\left(\frac{1}{2}\lim_{R\to\infty}\int_{\gamma_R} \frac{e^{iz}}{z^4+5z^2+1}\,dz\right)$$

$\cos(x) = \mathrm{Re}(e^{ix})$ $\forall x\in\mathbb{R}$

Note on β_R: $z = x+iy$ with $y>0$

$$\left|\frac{e^{iz}}{z^4+5z^2+1}\right| = \frac{|e^{ix}e^{-y}|}{|z^4+5z^2+1|} = \frac{e^{-y}}{|z^4+5z^2+1|} = \frac{e^{-y}}{|z|^4\left|1+\frac{5}{z^2}+\frac{1}{z^4}\right|}$$

for $|z|=R>3$, $\left|1+\frac{5}{z^2}+\frac{1}{z^4}\right| \geq 1-\frac{5}{|z|^2}-\frac{1}{|z|^4}$

$= 1-\frac{5}{R^2}-\frac{1}{R^4} > 1-\frac{5}{9}-\frac{1}{27}$

$> \frac{1}{3}$

$\therefore$ for $|z|=R>3$,

$$\left|\frac{e^{iz}}{z^4+5z^2+1}\right| < \frac{1}{R^4\left(\frac{1}{3}\right)}$$

$|z^4+5z^2+1|$
$\geq |z|^4-5|z|^2-1$
$= R^4-5R^2-1$

simpler.

So for $R>3$,

$$\left|\int_{\beta_R} \frac{e^{iz}}{z^4+5z^2+1}\,dz\right| \leq \int_{\beta_R} \left|\frac{e^{iz}}{z^4+5z^2+1}\right| |dz|$$

$$\leq \frac{1}{R^4\left(\frac{1}{3}\right)}\pi R \to 0 \text{ as } R\to\infty.$$

$$\therefore I = \frac{1}{2}\operatorname{Re} \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}\,dz}{z^4+5z^2+1}$$

$$= \frac{1}{2}\operatorname{Re} \lim_{R\to\infty} \int_{\gamma_R\cup\beta_R} \underbrace{\frac{e^{iz}}{z^4+5z^2+1}}_{N(z)}\,dz$$

$(z^2+1)(z^2+4)$

$$= \frac{1}{2}\operatorname{Re}\Big(2\pi i\big(\operatorname{Res}(N(z),i)+\operatorname{Res}(N(z),2i)\big)\Big)$$

$$N(z) = \frac{e^{iz}}{(z^2+1)(z^2+4)} = \frac{e^{iz}}{(z+i)(z-i)(z+2i)(z-2i)}$$

Both poles are simple:

$$\mathrm{Res}(N(z), i) = \frac{e^{i^2}}{(2i)(3i)(-i)} = \frac{e^{-1}}{6i}$$

$$\mathrm{Res}(N(z), 2i) = \frac{e^{i(2i)}}{(3i)(i)(4i)} = \frac{e^{-2}}{-12i}$$

$$\therefore I = \frac{1}{2}\,\mathrm{Re}\left(2\pi i\left(\frac{e^{-1}}{6i} - \frac{e^{-2}}{12i}\right)\right)$$

$$= \frac{1}{\cancel{2}}\cancel{2}\pi\left(\frac{e^{-1}}{6} - \frac{e^{-2}}{12}\right) = \boxed{\frac{\pi}{6}\left(\frac{1}{e} - \frac{1}{2e^2}\right)}$$

In book: see calculation of

$$\int_0^\infty \sin(x^2)\,dx = \sqrt{\frac{\pi}{8}}.$$

e^{iz^2}

Another application of the residue theorem.

$$f(z) = (z-z_0)^m g(z)$$

g: holom at z_0, $g(z_0) \neq 0$.

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1} g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)}.$$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}$$

$\frac{g'(z)}{g(z)}$: holomorphic near z_0

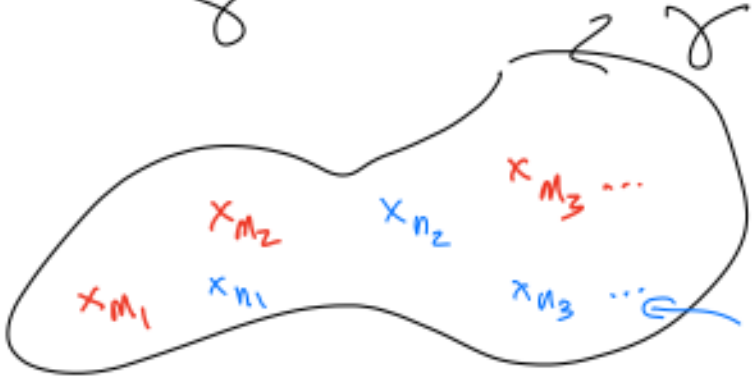
$$\therefore \int_\gamma \frac{f'(z)}{f(z)}\,dz = \underbrace{\int_\gamma \frac{m}{z-z_0}\,dz}_{=\,2\pi i m} + \underbrace{\int_\gamma \frac{g'(z)}{g(z)}\,dz}_{=\,0}$$

Generalize: Argument Principle: If f is meromorphic on $\operatorname{int}(\gamma)$ and on γ (no zeros or poles on γ)

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\big((n_1+n_2+\cdots) - (m_1+m_2+\cdots)\big)$$

$$= 2\pi i\,(Z_f - P_f)$$

$Z_f = $ # of zeros of f
$P_f = $ # of poles
counting multiplicities



poles of order m_j

zeros of order n_j